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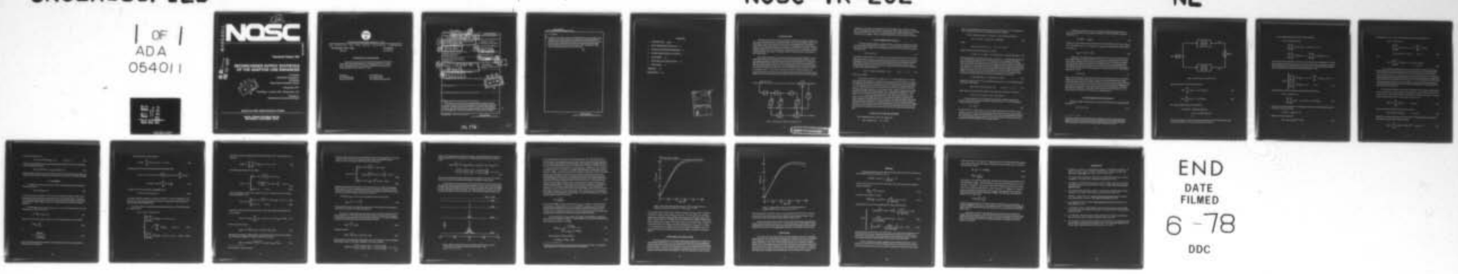

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SECOND-ORDER OUTPUT STATISTICS OF THE ADAPTIVE LINE ENHANCER

JT Rickard
ORINCON Corporation
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The adaptive line enhancer (ALE) is an adaptive digital filter designed to suppress uncorrelated components of its input, while passing any narrowband components with little attenuation. The purpose of this paper is to analyze the second-order output statistics of the ALE in steady-state operation, for input samples consisting of weak narrowband signals in white Gaussian noise. The ALE output is shown to be the sum of two uncorrelated

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components, one arising from optimum finite-lag Wiener filtering of the narrowband input components, and the other arising from the misadjustment error associated with the adaptation process. General expressions are given for the output autocorrelation function and power spectrum with arbitrary narrowband input signals, and the case of a single sinusoid in white noise is worked out as an example. Finally, the significance of these results to practical applications of the ALE is mentioned.

CONTENTS

1 INTRODUCTION

The adaptive line enhancer (ALE) is an adaptive digital filter designed to separate its input into two components, one consisting primarily of any narrowband signals present in the input (hence the term "line enhancer"), and the other consisting primarily of broadband noise, which is always assumed present in the input. A block diagram of the device is shown in figure 1, where the above components are denoted by r(k) and $\epsilon(k)$, respectively. Since the detailed operation of the ALE has previously been described [1–4], we shall only summarize its basic properties in this paper, and propose a model of the filter which is convenient for our further analysis.

The purpose of this paper is to analyze the second-order output statistics, i.e., autocorrelation function and power spectrum, of the ALE during steady-state operation, with input samples consisting of weak narrowband signals in white Gaussian noise. "Steady-state operation" implies that the ALE has a stationary input, and has processed enough data so that all start-up transients have died out. The only assumption made on the spectral shape of the narrowband components is that their inverse bandwidths be large with respect to the filter sampling period f_s^{-1}. The results are expressed in terms of the ALE parameters, the second-order statistics of the input, and the optimum (finite-lag) Wiener filter weights for the problem at hand. It is shown that the ALE output sequence r(k) consists of two uncorrelated

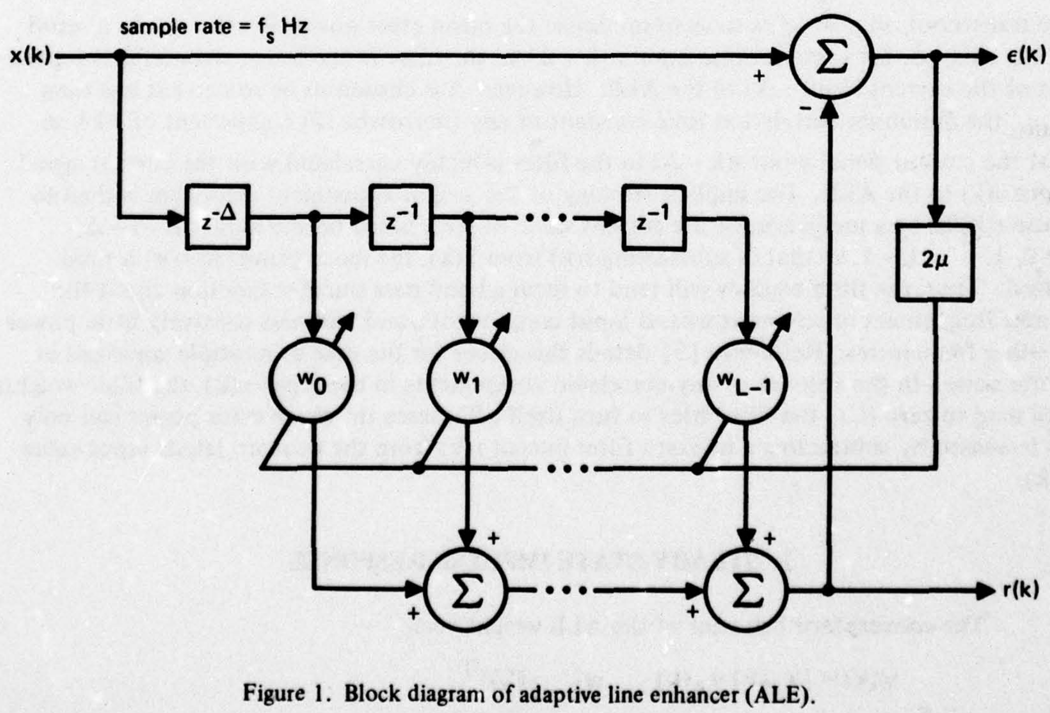

Figure 1. Block diagram of adaptive line enhancer (ALE).

components: one of them corresponding to the Wiener filtered input, and the other being the result of passing the input through a "misadjustment filter" whose properties are described. The case where the input consists of a deterministic sinusoid in white Gaussian noise is worked out as an example. Finally, a number of applications for these results are described.

2 BASIC PROPERTIES OF THE ALE

The block diagram of figure 1 reveals the ALE to be a particular variation of an adaptive noise canceler [5], wherein the "reference" or "desired" response is merely the current input x(k), and the "primary" input is a delayed version of x(k). The input x(k) is assumed to be of the form

$$x(k) = s(k) + n(k), \tag{1}$$

where s(k) is, in general, the sum of a number of narrowband components having nonoverlapping power spectra, and n(k) is a zero-mean white Gaussian sequence with power ν^2 which is independent of s(k). In the lower channel, the delayed input is passed through an adaptive linear transversal filter whose output r(k) is then subtracted from x(k) to form an error sequence $\epsilon(k)$. This error sequence is fed back to adjust the filter weights $w_i(k)$ according to the Widrow-Hoff LMS algorithm [5],

$$w_i(k+1) = w_i(k) + 2\mu\epsilon(k)\,x(k-i-\Delta), \qquad i = 0, 1, \ldots, L-1, \tag{2}$$

where μ is a constant.

The principle of operation is as follows [1, 2]: the weight algorithm in (2) adjusts the transversal filter weights so as to minimize the mean error power $E\{\epsilon(k)^2\}$. As a result of the delay Δ, the current noise input $n(k-\Delta)$ to the filter in the lower channel is independent of the current input x(k) to the ALE. However, Δ is chosen to be somewhat less than τ_{min}, the minimum correlation time constant of any (narrowband) component of s(k), so that the current signal input $s(k-\Delta)$ to the filter is highly correlated with the current signal input s(k) to the ALE. The implicit strategy of the weight adjustment algorithm is then to cause r(k) to be a prediction of the current value of s(k), based on the data $x(k-i-\Delta)$, $i = 0, 1, \ldots, L-1$, so that in subtracting r(k) from x(k), the mean power in $\epsilon(k)$ is minimized. Thus, the filter weights will tend to form a band-pass transfer function about the center frequencies of any narrowband input components, and will pass relatively little power at other frequencies. Reference [3] details this effect for the case of multiple sinusoids in white noise. In the absence of any correlated components in the input x(k), the filter weights will tend to zero (i.e., the filter tries to turn itself off), since the mean error power can only be increased by subtracting a nonzero filter output r(k) from the (uncorrelated) input value x(k).

3 STEADY-STATE IMPULSE RESPONSE

The convergence behavior of the ALE weight vector

$$\underline{w}(k) = [w_0(k)\; w_1(k) \ldots w_{L-1}(k)]^T$$

under various assumptions has been discussed by previous authors [1, 4]. For purposes of clarity, we shall briefly discuss the assumptions made in our analysis.

The ALE weight vector $\underline{w}(k+1)$ at time $k+1$ may be written from (2) as

$$\underline{w}(k+1) = \underline{w}(k) + 2\mu\, \underline{x}(k)\, [x(k) - \underline{x}^T(k)\, \underline{w}(k)], \tag{3}$$

where

$$\underline{x}(k) \triangleq [x(k-\Delta)\; x(k-\Delta-1) \ldots x(k-\Delta-L+1)]^T.$$

Taking the expectation of both sides of (3), we have

$$E\,\{\underline{w}(k+1)\} = E\,\{\underline{w}(k)\} + 2\mu\, [E\,\{\underline{x}(k)\, x(k)\} - E\,\{\underline{x}(k)\, \underline{x}^T(k)\, \underline{w}(k)\}]. \tag{4}$$

Since $\underline{w}(k)$ depends on all of the past data $x(k-1), x(k-2), \ldots$, the last expectation in the above equation is not separable, and an exact analysis of the convergence of $E\,\{\underline{w}(k)\}$ becomes intractable beyond this point. It is clear from (3), however, that for very small values of μ (relative to the variance of $x(k)$), the weight vector $\underline{w}(k)$ changes very slowly with time. Thus, in the steady-state circumstances considered here, one would expect little statistical dependence of the weight vector on the relatively brief span of data (L samples) encompassed by the vector $\underline{x}(k)$. This would particularly be so in the low signal-to-noise ratio (SNR) case of interest, since most of the power in $x(k)$ is then due to white noise. Hence, we make the assumption that $\underline{w}(k)$ is statistically independent of $\underline{x}(k)$, so that

$$E\,\{\underline{x}(k)\, \underline{x}^T(k)\, \underline{w}(k)\} = E\,\{\underline{x}(k)\, \underline{x}^T(k)\}\; E\,\{\underline{w}(k)\}. \tag{5}$$

Letting Φ and $\underline{d}$ denote the $L \times L$ autocorrelation matrix and $L \times 1$ cross-correlation vector, respectively, with elements

$$\Phi_{pq} = \varphi_x(p-q) \text{ and } d_p = \varphi_x(p+\Delta), \qquad p,q = 0, 1, \ldots, L-1, \tag{6}$$

where $\varphi_x(\ell)$ is the autocorrelation function of the input $x(k)$, we may then write (4) as

$$E\,\{\underline{w}(k+1)\} = E\,\{\underline{w}(k)\} + 2\mu\, [\underline{d} - \Phi\, E\,\{\underline{w}(k)\}]. \tag{7}$$

It has been shown in [5] and elsewhere that the above difference equation for $E\,\{\underline{w}(k)\}$ converges (starting from an arbitrary initial value) as $k \to \infty$ to $\underline{w}^*$, the optimum Wiener weight vector given by

$$\underline{w}^* = \Phi^{-1}\, \underline{d}, \tag{8}$$

provided that $0 < \mu < \lambda_{max}^{-1}$, where λ_{max} is the largest eigenvalue of the matrix Φ. Also, a number of authors [1-5] have reported experimental evidence which supports the validity of the assumption (5), leading to the expression (8). Hence, insofar as this assumption is valid, the mean ALE impulse response function in steady-state operation is identical to the corresponding optimum finite-lag Wiener filter impulse response.

Under the assumptions of small μ, low SNR, and white Gaussian input noise, an approximate expression for the steady-state covariance of the ALE weight vector $\underline{w}(k)$ is derived in [1] as

$$\text{cov}\ \{\underline{w}(k)\} = \mu \xi_{min} I, \tag{9}$$

where ξ_{min} is the minimum mean-squared error associated with Wiener filtering of the input, and I is the identity matrix.[1] An expression for ξ_{min} may be obtained from classical Wiener filter theory as

$$\xi_{min} = E\ \{x(k)^2\} - \underline{d}^T \underline{w}^*. \tag{10}$$

In summary, the steady-state impulse response of the ALE is a very slowly varying vector random process with mean value $\underline{w}^*$ as in (8), and covariance given by (9) and (10). By assuming a sufficiently small value of μ, the weight vector may be considered constant over time intervals which are large relative to the length (in time) of the transversal filter. This suggests the model shown in figure 2 for the ALE structure during steady-state operation. The ALE weight vector $\underline{w}$ is decomposed into its mean value $\underline{w}^*$ and a "misadjustment weight vector" $\underline{\widetilde{w}}$, viz.,

$$\underline{w} = \underline{w}^* + \underline{\widetilde{w}}, \tag{11}$$

where we have dropped the time index, due to the near-constant behavior of $\underline{w}$. The Wiener weight vector $\underline{w}^*$ is a deterministic quantity given by (8), and the misadjustment weight vector $\underline{\widetilde{w}}$ (so named because it represents a random bias away from $\underline{w}^*$) is a zero-mean random vector, with covariance given by (9) and (10), which is assumed statistically independent of the current data in the filter. Thus, the ALE output r(k) is the superposition of the Wiener filter (WF) and misadjustment filter (MF) outputs when excited by the input x(k). This model will prove convenient in the analysis of the following section.

4 SECOND-ORDER OUTPUT STATISTICS

Referring to figure 2 and equation (11), the ALE output r(k) may be expressed mathematically as

$$r(k) = r^*(k) + \widetilde{r}(k), \tag{12}$$

---

[1] *It has been shown in [4] that a more precise expression for cov $\{w(k)\}$ may be obtained for certain special inputs, in particular that of a sinusoid in white noise, which is used as an example in section 4. The expression in [4] contains an added term to that in (9). However, we show in the appendix that the effect of this term is of minor significance. Hence we shall employ (9) as the weight vector covariance throughout the main body of the paper.*

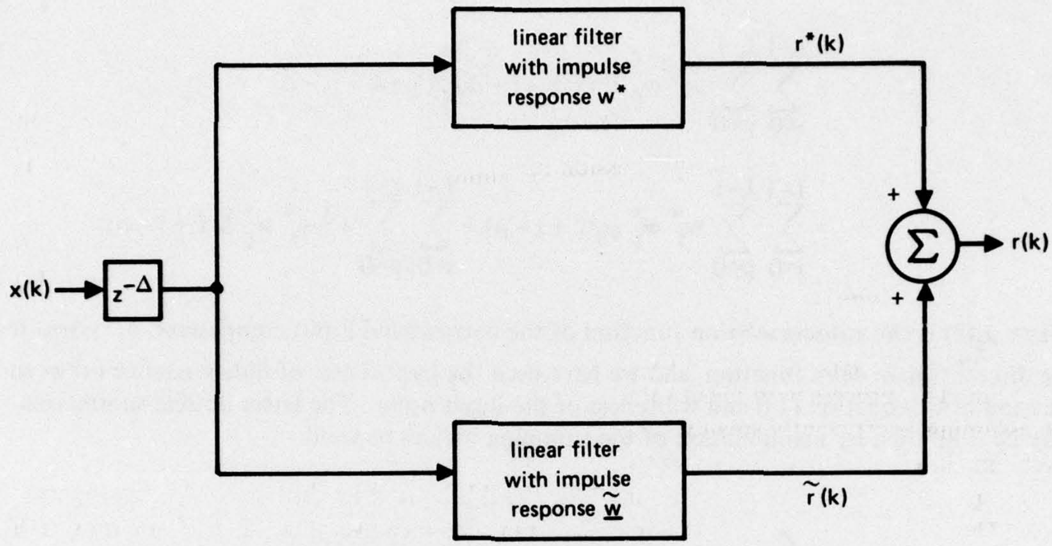

Figure 2. Equivalent model for steady-state ALE.

where the sequences $r^*(k)$ and $\tilde{r}(k)$ are given by

$$r^*(k) = \sum_{i=0}^{L-1} w_i^* x(k-i-\Delta) = \underline{x}^T(k)\,\underline{w}^*, \tag{13}$$

$$\tilde{r}(k) = \sum_{i=0}^{L-1} \tilde{w}_i\, x(k-i-\Delta) = \underline{x}^T(k)\,\underline{\tilde{w}}. \tag{14}$$

The cross-correlation between $r^*(k)$ and $\tilde{r}(k)$ is

$$\begin{aligned} E\{r^*(k)\tilde{r}(\ell)\} &= E\{\underline{\tilde{w}}^T\, \underline{x}(k)\, \underline{x}^T(k)\, \underline{w}^*\} \\ &= E\{(\underline{w} - \underline{w}^*)\, \underline{x}(k)\, \underline{x}^T(k)\}\, \underline{w}^* \\ &= 0 \end{aligned}$$

by virtue of equation (5). Thus, the second-order statistics of r(k) will be the sum of the second-order statistics of $r^*(k)$ and $\tilde{r}(k)$, which we now derive.

The autocorrelation function $\varphi^*(\ell)$ of $r^*(k)$ is given by

$$\begin{aligned}\varphi^*(\ell) &= E\{r^*(k)\, r^*(k+\ell)\}\\ &= \sum_{i=0}^{L-1}\sum_{p=0}^{L-1} w_i^* w_p^* E\{x(k-i-\Delta)\, x(k+\ell-p-\Delta)\}\\ &= \sum_{i=0}^{L-1}\sum_{p=0}^{L-1} w_i^* w_p^* \varphi_s(\ell+i-p) + \sum_{i=0}^{L-1}\sum_{p=0}^{L-1} \nu^2 w_i^* w_p^* \delta(\ell+i-p),\end{aligned}$$

where $\varphi_s(\ell)$ is the autocorrelation function of the narrowband input component, $\delta(\cdot)$ denotes the discrete Dirac delta function, and we have used the hypotheses of independence between s(k) and n(k) (equation (1)) and whiteness of the input noise. The latter double summation may be simplified by manipulation of the summing indices to yield

$$\varphi^*(\ell) = \begin{cases} \displaystyle\sum_{i=0}^{L-1}\sum_{p=0}^{L-1} w_i^* w_p^* \varphi_s(\ell+i-p) + \nu^2 \sum_{i=0}^{L-|\ell|-1} w_i^* w_{i+|\ell|}^*, & |\ell| \leqslant L-1,\\ \displaystyle\sum_{i=0}^{L-1}\sum_{p=0}^{L-1} w_i^* w_p^* \varphi_s(\ell+i-p), & |\ell| \geqslant L. \end{cases} \tag{15}$$

In similar fashion, the autocorrelation function $\tilde{\varphi}(\ell)$ of $\tilde{r}(k)$ is given by

$$\begin{aligned}\tilde{\varphi}(\ell) &= E\{\tilde{r}(k)\, \tilde{r}(k+\ell)\}\\ &= \sum_{i=0}^{L-1}\sum_{p=0}^{L-1} [\varphi_s(\ell+i-p) + \nu^2 \delta(\ell+i-p)]\; E\{\tilde{w}_i \tilde{w}_p\},\end{aligned} \tag{16}$$

where we have used the assumption of independence between the MF weights and the data in the filter. From (9), however,

$$E\{\tilde{w}_i \tilde{w}_p\} = \mu\, \xi_{min}\, \delta(i-p),$$

and thus (16) may be reduced to

$$\tilde{\varphi}(\ell) = \mu\, \xi_{min}\, L[\varphi_s(\ell) + \nu^2\, \delta(\ell)]\,. \tag{17}$$

Combining (15) and (17), the autocorrelation function $\varphi_r(\ell)$ of r(k) is given by

$$\varphi_r(\ell) = E\{r(k)\, r(k+\ell)\}$$

$$= \sum_{i=0}^{L-1} \sum_{p=0}^{L-1} w_i^* w_p^* \varphi_s(\ell + i - p) + \nu^2 \sum_{i=0}^{L-|\ell|-1} w_i^* w_{i+|\ell|}^*$$

$$+ \mu\, \xi_{min}\, L[\varphi_s(\ell) + \nu^2\, \delta(\ell)], \qquad |\ell| \leqslant L - 1, \tag{18a}$$

and

$$\varphi_r(\ell) = \sum_{i=0}^{L-1} \sum_{p=0}^{L-1} w_i^* w_p^* \varphi_s(\ell + i - p) + \mu\, \xi_{min}\, L[\varphi_s(\ell)], \qquad |\ell| \geqslant L. \tag{18b}$$

It is worthwhile noting that the first two terms in equation (18a) are the result of Wiener filtering of the input signal, while the last term (which is due to the misadjustment noise filtering of the input) is a scaled copy of the input autocorrelation function $\varphi_x(\ell)$. Thus, the average effect of the misadjustment noise in the ALE is to produce an output component having the same (scaled) autocorrelation function as the input x(k), yet uncorrelated with the Wiener filter output signal. Since $\mu\, \xi_{min}\, L \ll 1$ in most applications, this MF output will be greatly attenuated with respect to the input. This provides a further illustration of the line enhancement property of the ALE, since the MF output is the only component of r(k) which contains white noise.

The power spectrum $P_r(\omega)$ of r(k) may be obtained by direct Fourier transformation of $\varphi_r(\ell)$. Alternatively, one may obtain the respective power spectra $P^*(\omega)$ and $\widetilde{P}(\omega)$ separately as follows. Since the WF weight vector $\underline{w}^*$ defines a discrete linear filter, we may compute its transfer function $H^*(e^{j\omega})$ on the unit circle as [6, p. 21]

$$H^*(e^{j\omega}) = \sum_{p=0}^{L-1} w_p^*\, e^{-j\omega p}, \qquad -\pi \leqslant \omega \leqslant \pi. \tag{19}$$

Then, the power spectrum $P^*(\omega)$ is given by [6, p. 393]

$$P^*(\omega) = |H^*(e^{j\omega})|^2\, P_x(\omega), \qquad -\pi \leqslant \omega \leqslant \pi, \tag{20}$$

where $P_x(\omega)$ is the power spectrum of the input x(k). This latter quantity is the Fourier transform of the input autocorrelation function:

$$P_x(\omega) = \sum_{\ell=-\infty}^{\infty} [\varphi_s(\ell) + \nu^2\, \delta(\ell)]\, e^{-j\omega\ell}, \quad = P_s(\omega) + \nu^2. \tag{21}$$

Hence, from (20) and (21),

$$P^*(\omega) = |H^*(e^{j\omega})|^2\ [P_s(\omega) + \nu^2], \qquad -\pi \leqslant \omega \leqslant \pi. \tag{22}$$

The power spectrum $\widetilde{P}(\omega)$ may be obtained by inspection from (17) and (21). Combining these two expressions we have

$$P_r(\omega) = [|H^*(e^{j\omega})|^2 + \mu\, \xi_{min}\, L]\ [P_s(\omega) + \nu^2]. \tag{23}$$

Again, we observe that the output of the ALE consists of the Wiener filtered input signal plus an uncorrelated component having the same power spectrum (except for scaling) as the input.

5 AN EXAMPLE

To illustrate the results of the previous section, we shall consider the case where the ALE input is given by

$$s(k) = A \sin(\omega_0\, k + \theta), \tag{24}$$

where A, ω_0, and θ are constants. This of course is a nonstationary signal. However, with the exception of the Wiener filter output replica of s(k), the remaining output components in r(k) are approximately stationary, and thus we may compute their stationary second-order output statistics. These results are shown to be compatible with the general case analyzed previously.

The WF weights for this case are shown in [2, 3] to have the form (when ω_0 is several multiples of π/L away from zero or π),

$$w_i^* = \frac{2a^*}{L} \cos \omega_0(i + \Delta), \tag{25}$$

where Δ is the delay value shown in figure 1, and a^* is defined in terms of the input SNR,

$$SNR_{in} = \frac{A^2}{2\nu^2}, \tag{26}$$

by

$$a^* = \frac{\left(\frac{L}{2}\right) SNR_{in}}{1 + \left(\frac{L}{2}\right) SNR_{in}}. \tag{27}$$

Note that the WF weights do not depend on the input signal phase θ, thus enabling the stationary analysis which follows.

The expected value of r(k) is given as

$$E\{r(k)\} = \sum_{i=0}^{L-1} w_i^* \, A \sin\left[\omega_0 (k - i - \Delta) + \theta\right]. \tag{28}$$

Substituting (25) into (28) and using trigonometric identities yields

$$E\{r(k)\} = a^* A \sin(\omega_0 k + \theta) + \frac{a^* A}{L}\Bigg\{\sin\left[\omega_0 (k - 2\Delta) + \theta\right] \sum_{i=0}^{L-1} \cos 2\omega_0 i$$

$$- \cos\left[\omega_0 (k - 2\Delta) + \theta\right] \sum_{i=0}^{L-1} \sin 2\omega_0 i\Bigg\}. \tag{29}$$

For $\omega_0 L \gg \pi$, the second term above is negligible; hence

$$E\{r(k)\} \cong a^* A \sin(\omega_0 k + \theta). \tag{30}$$

This result is intuitively satisfying: note from (27) that $a^* \cong 1$ when $(L/2)\ SNR_{in} \gg 1$, and thus the mean ALE output is approximately equal to the sinusoidal component present in the input when adequate SNR_{in} exists.

Let $\varphi^*(k, m)$ be the (nonstationary) autocorrelation function of $r^*(k)$ for this example. Then from (15),

$$\varphi^*(k, k + \ell) = E\{r^*(k)\, r^*(k + \ell)\}$$

$$= \begin{cases} \displaystyle\sum_{i=0}^{L-1} \sum_{p=0}^{L-1} w_i^* w_p^* \varphi_s(k - i - \Delta, k + \ell - p - \Delta) \\ \qquad \displaystyle + \nu^2 \sum_{i=0}^{L-|\ell|-1} w_i^* w_{i+|\ell|}^*, & |\ell| \leq L - 1, \quad (31a) \\ \displaystyle\sum_{i=0}^{L-1} \sum_{p=0}^{L-1} w_i^* w_p^* \varphi_s(k - i - \Delta, k + \ell - p - \Delta), & |\ell| \geq L, \quad (31b) \end{cases}$$

where $\varphi_s(k, m)$ is the nonstationary autocorrelation function of s(k). Letting $\varphi_s^*(k, m)$ be defined by

$$\varphi_s^*(k, m) = \sum_{i=0}^{L-1} \sum_{p=0}^{L-1} w_i^* w_p^* \varphi_s(k - i - \Delta, m - p - \Delta), \tag{32}$$

and substituting (25) into (31a), we obtain

$$\varphi^*(k, k+\ell) = \begin{cases} \varphi_s^*(k, k+\ell) + \dfrac{2a^{*2}\nu^2}{L^2}\left[(L - |\ell|)\cos\omega_0\ell \right. \\ \qquad \left. + \dfrac{\cos\omega_0(L+1)}{\sin\omega_0}\sin\omega_0(L - |\ell|)\right], & |\ell| \leqslant L - 1, \quad (33a) \\ \varphi_s^*(k, k+\ell), & |\ell| \geqslant L, \quad (33b) \end{cases}$$

as the autocorrelation function $r^*(k)$. Again, for $\omega_0 L \gg \pi$, only the first term within the brackets is significant; hence

$$\varphi^*(k, k+\ell) \cong \begin{cases} \varphi_s^*(k, k+\ell) + \dfrac{2a^{*2}\nu^2}{L^2}(L - |\ell|)\cos\omega_0\ell, & |\ell| \leqslant L - 1, \quad (34a) \\ \varphi_s^*(k, k+\ell), & |\ell| \geqslant L. \quad (34b) \end{cases}$$

Now let $\tilde{\varphi}(k, m)$ be the autocorrelation function of $\tilde{r}(k)$. By analogy with equations (16)-(17), we obtain[2]

$$\tilde{\varphi}(k, k+\ell) = \mu\,\xi_{min} \sum_{i=0}^{L-1} \varphi_s(k - i - \Delta, k + \ell - i - \Delta) + \mu\,\xi_{min}\,\nu^2 L\delta(\ell). \tag{35}$$

However, from (24) we have

$$\varphi_s(k, k+\ell) = \frac{A^2}{2}\left[\cos\omega_0\ell - \cos(2\omega_0 k + \omega_0\ell + 2\theta)\right], \tag{36}$$

and when this expression is substituted into (35) and summed, the sum over the nonstationary second term is negligible by comparison with the first. Hence,

$$\tilde{\varphi}(k, k+\ell) \cong \tilde{\varphi}(\ell) \triangleq \frac{\mu\,\xi_{min} A^2 L}{2}\cos\omega_0\ell + \mu\,\xi_{min}\,\nu^2 L\delta(\ell), \tag{37}$$

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[2] *See the appendix for a more precise expression.*

where the double argument in the autocorrelation function is no longer needed. Thus, from equations (34a, b) and (37), the steady-state ALE output autocorrelation function $\varphi_r(k, k+\ell)$ with a deterministic sinusoid plus white noise input is

$$\varphi_r(k, k+\ell) = \begin{cases} \varphi_s^*(k, k+\ell) + \dfrac{2a^{*2}\nu^2}{L^2}(L - |\ell|)\cos\omega_0\ell \\ \quad + \mu\,\xi_{min}\,L\left[\dfrac{A^2}{2}\cos\omega_0\ell + \nu^2\,\delta(\ell)\right], & |\ell| \leq L-1, \quad (38a) \\ \varphi_s^*(k, k+\ell) + \mu\,\xi_{min}\,L\dfrac{A^2}{2}\cos\omega_0\ell, & |\ell| \geq L. \quad (38b) \end{cases}$$

This particular result was first derived in [7], and later in [8], by a direct, but somewhat tedious, method. The above equation verifies that for a sinusoid in white noise, all of the ALE output components except the Wiener filtered signal output are approximately stationary when ω_0 is several multiples of π/L away from zero or π.

The value of ξ_{min} in this case is easily computed from (10) and (25) as

$$\xi_{min} \cong \nu^2 + (1 - a^*)\frac{A^2}{2}. \tag{39}$$

The approximation arises from neglecting summations over the argument of nonstationary cosine terms, as was done in arriving at equation (37).

We shall now compute the power spectrum of r(k) in the manner of equation (23) of the preceding section. The nonstationary input signal (24) may be dealt with by assuming that the phase θ of the sinusoid is a uniform random variable on $[-\pi, \pi]$, thereby resulting in a stationary sinusoidal input signal with autocorrelation function

$$\varphi_s(\ell) = \frac{A^2}{2}\cos\omega_0\ell, \tag{40}$$

and power spectrum

$$P_s(\omega) = \frac{\pi A^2}{2}\,[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]. \tag{41}$$

The only effect of this random phase assumption on the ALE output is to convert the Wiener filtered signal output to a stationary component. From (25), we obtain

$$|H^*(e^{j\omega})|^2 = \frac{a^{*2}}{L^2}\left[\frac{1 - \cos L(\omega - \omega_0)}{1 - \cos(\omega - \omega_0)} + \frac{1 - \cos L(\omega + \omega_0)}{1 - \cos(\omega + \omega_0)}\right], \quad -\pi \leq \omega \leq \pi, \tag{42}$$

where we have neglected the cross-product of positive- and negative-frequency terms in the magnitude-squared Fourier transform of (25). Substituting (41) and (42) into (23) then yields

$$P_r(\omega) = \frac{\pi A^2}{2} \left(a^{*2} + \mu\, \xi_{min} L\right) \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] + \mu\, \xi_{min}\, \nu^2 L$$

$$+ \frac{a^{*2} \nu^2}{L^2} \left[\frac{1 - \cos L(\omega - \omega_0)}{1 - \cos(\omega - \omega_0)} + \frac{1 - \cos L(\omega + \omega_0)}{1 - \cos(\omega + \omega_0)}\right], \quad -\pi \leqslant \omega \leqslant \pi. \tag{43}$$

One may verify that this expression is identical to that obtained by direct Fourier transformation of (38a, b) with the input signal stationarity assumption discussed above.

As described in the previous section, the delta functions in (43) arise from the linear filtering of the input sinusoid by the WF and MF weights, the white noise term from passing the input white noise through the MF weights, and the narrowband noise term from filtering the input white noise with the WF weights. The relative magnitudes of the latter two terms as a function of SNR_{in} are illustrated in figure 3. Note that the narrowband output noise

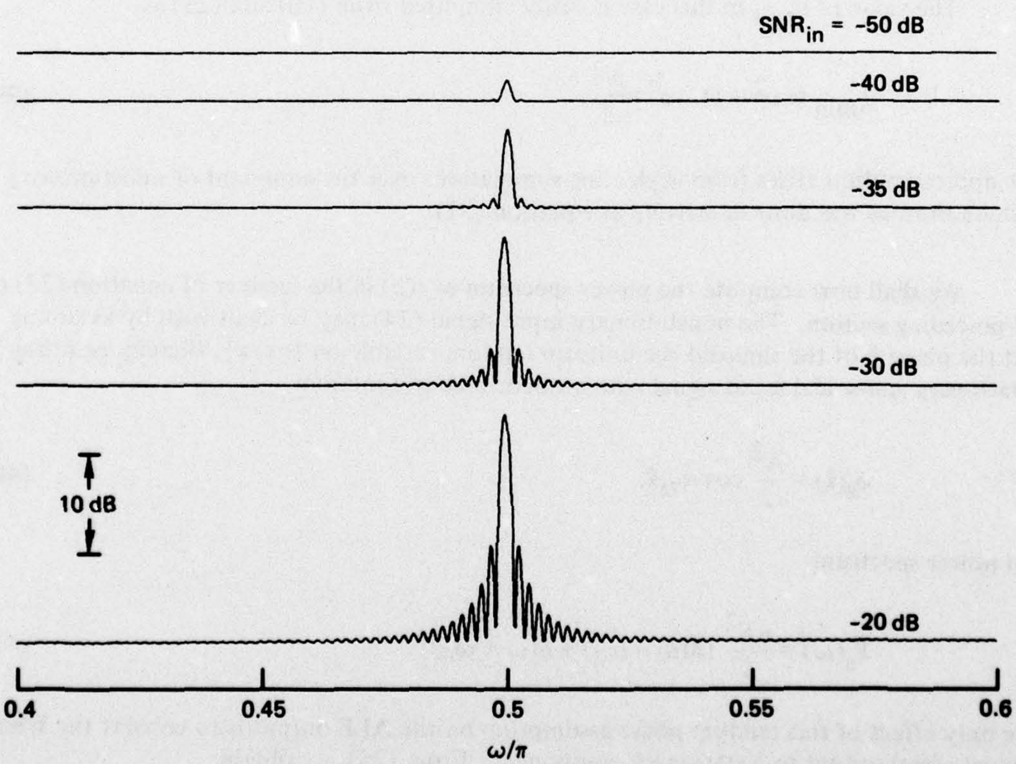

Figure 3. Plots of continuous components of ALE output power spectrum as a function of SNR_{in} (flat portions of all spectra have same absolute level); L = 1024, $\mu\,\xi_{min} = 4 \times 10^{-6}$, $\omega_0 = 0.5\,\pi$.

clearly dominates the white output noise as SNR_{in} increases. Since this white noise represents the bulk of the misadjustment noise, these plots demonstrate that the ALE produces a very respectable approximation to the Wiener filter, with no a priori knowledge of the input statistics, when adequate SNR_{in} exists. For smaller values of SNR_{in}, the attenuation factor a^* in the WF transfer function $H^*(e^{j\omega})$ becomes significant, as indicated by (42). From (27), we observe that $0 \leqslant a^* < 1$, and a^* is a nonlinear, monotonically increasing function of the product $L \cdot SNR_{in}$. If this latter quantity does not greatly exceed unity, then a^* is somewhat less than unity, and the WF transfer function is attenuated. This result is a property of finite-length Wiener filtering, and is a consequence of the minimum error power criterion. The finite-length WF can reduce the narrowband filtered noise power within the filter pass-band only by reducing the magnitude of the transfer function at ω_0. This simultaneously reduces the filtered signal amplitude, thereby increasing the error power resulting from incomplete signal cancellation. The minimum total error power is achieved at a compromise value of a^* (< 1), and this can significantly affect estimates of the sinusoid amplitude (obtained by spectrum analysis of the ALE output) when SNR_{in} is too small. Increasing the filter length L will narrow the filter pass-band about the center frequency,[3] thus improving the estimate of signal amplitude for a given SNR_{in}. To illustrate the behavior of a^*, we have plotted in figure 4 (solid curve) the ratio

$$R \triangleq \frac{a^{*2}}{\mu \, \xi_{min} \, L}\ , \tag{44}$$

as a function of SNR_{in}, using the same value of $\mu \, \xi_{min} \, L$ as in figure 3. This curve demonstrates both the nonlinear behavior of a^*, and the relative magnitudes of the WF and MF sinusoidal output components, as functions of SNR_{in}. Note that for SNR_{in} greater than -35 dB, most of the sinusoidal output power is due to the WF signal output, but SNR_{in} must be greater than -20 dB in order to obtain a reasonable amplitude estimate.

As a final illustration of the results in the example, it is interesting to examine the gain of the ALE as a function of SNR_{in}. The output SNR of the ALE may be obtained from (30) and (38a) by setting $\ell = 0$, yielding

$$SNR_{out} = \frac{a^{*2}\, SNR_{in}}{\dfrac{2a^{*2}}{L} + \mu \, \xi_{min} \, L\,(1 + SNR_{in})}\ . \tag{45}$$

The ALE gain G (in dB) is defined by

$$G \triangleq SNR_{out} - SNR_{in} \quad \text{(dB)}, \tag{46}$$

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[3] *Increasing L also raises the misadjustment noise level, but this can be compensated by decreasing μ, i.e., lengthening the initial convergence time. For a steady-state analysis, this latter time is unimportant.*

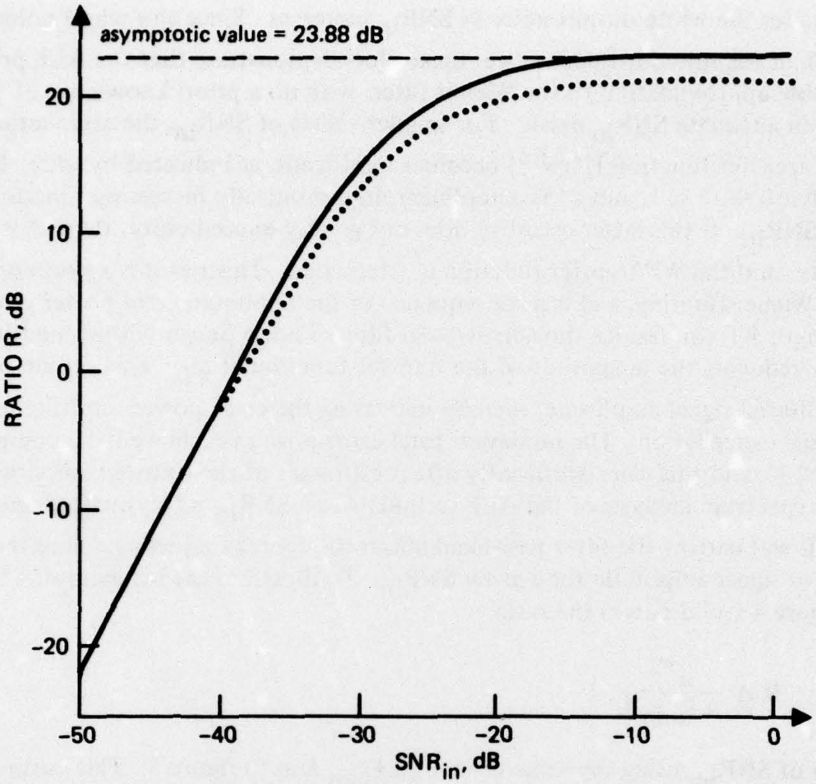

Figure 4. Ratio of WF to MF sinusoidal output powers; solid curve given by (44); see appendix for explanation of dotted curve; L = 1024, $\mu\xi_{min} = 4 \times 10^{-6}$.

and is plotted in figure 5 (solid curve), using the same values of $\mu\ \xi_{min}$ and L as in figure 3. Note that for very low values of SNR_{in}, the quantity a^{*2} in (45) is near zero, resulting in negative gain. Physically speaking, the MF output dominates the WF signal output in this range, as shown in figure 3. As SNR_{in} increases, the gain becomes positive, reaching a peak value of approximately 22 dB, and then drops steadily with increasing SNR_{in}. This behavior is obvious from (45), since SNR_{out} approaches a limiting value of $(\mu\ \xi_{min}\ L)^{-1}$ as $SNR_{in} \to \infty$. ***This limiting value of SNR_{out} is due to the unavoidable misadjustment noise of the ALE.*** **Observe, however, the wide and useful range of SNR_{in} over which the ALE gain is positive.**

6 DISCUSSION AND APPLICATIONS

The primary significance of our results is their general applicability to arbitrary narrowband input components; i.e., one need not go through the tedious analysis required to directly compute the second-order output statistics of the ALE for each narrowband signal model of interest. The burden of the analysis is now placed on finding the Wiener filter solution for the problem at hand. This is generally an easier problem, and is in any event a prerequisite for the direct method of computation. Once this is obtained, one may then write

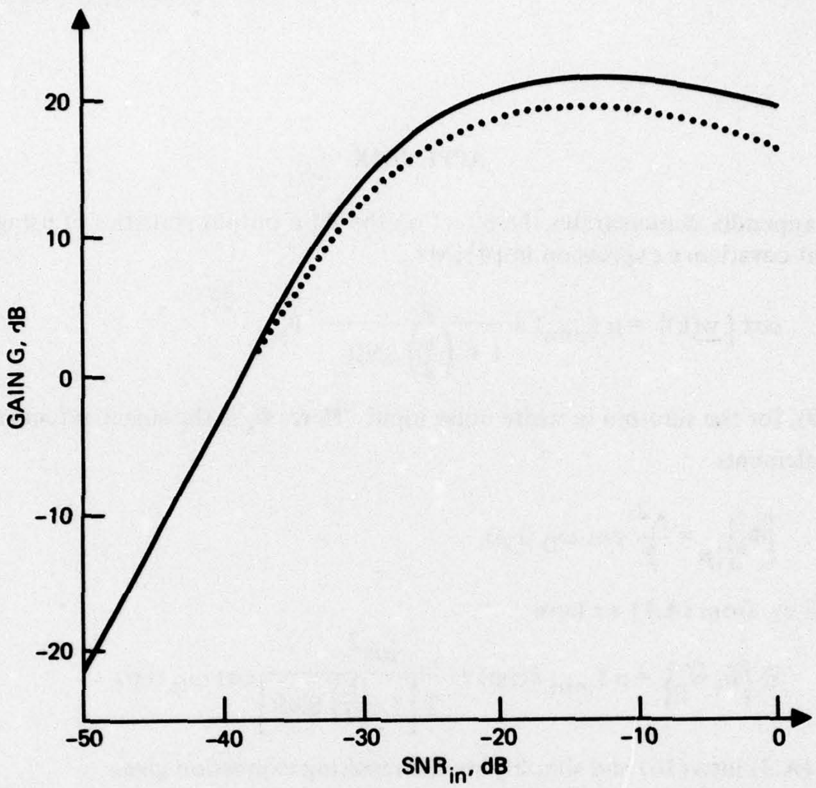

Figure 5. ALE gain; solid curve given by (45) and (46); see appendix for explanation of dotted curve; L = 1024, $\mu \xi_{min} = 4 \times 10^{-6}$.

down $\varphi_r(\ell)$ and $P_r(\omega)$ virtually by inspection, using equations (18a, b) and (23). The cases for which the Wiener filter solution has been obtained include multiple sinusoids in white noise [3] and a class of narrowband random signals in white noise [9].

Applications of the ALE for which our results will be useful include narrowband signal detection and encoding of noisy sources. In both applications, the ALE might be used in a prefiltering role, and hence the output power spectrum is of key importance to the further analysis of such systems. Forthcoming papers will discuss these problems in greater detail.

7 CONCLUSION

The steady-state ALE impulse response has been modeled as the sum of two parallel impulse responses, one of them being the optimum finite-lag Wiener filter for the narrowband input components, and the other being a misadjustment noise filter with random impulse response coefficients. The outputs of these parallel channels are uncorrelated. Using this model, we have derived general expressions for the ALE output autocorrelation function and power spectrum for an input consisting of narrowband signals in white noise. An example is given for the case of a single sinusoid in white noise. These results are useful in a number of ALE applications, including narrowband signal detection and noisy-source encoding.

APPENDIX

This appendix demonstrates the effect on the ALE output statistics of using the more precise weight covariance expression in [4], viz.,

$$\mathrm{cov}\left\{\underline{w}(k)\right\} = \mu\, \xi_{min} I + \frac{\mu}{1 + \left(\frac{L}{2}\right) SNR}\, \Phi_s\, , \tag{A.1}$$

in place of (9), for the sinusoid in white noise input. Here, Φ_s is the signal autocorrelation matrix with elements

$$\left[\Phi_s\right]_{ip} = \frac{A^2}{2} \cos \omega_0 (i-p), \tag{A.2}$$

as in (40). Thus, from (A.1) we have

$$E\left\{\widetilde{w}_i\, \widetilde{w}_p\right\} = \mu\, \xi_{min}\, \delta(i-p) + \frac{\mu A^2}{2\left[1 + \left(\frac{L}{2}\right) SNR\right]} \cos \omega_0 (i-p)\ . \tag{A.3}$$

Substituting (A.3) into (16) and simplifying the resulting expression gives

$$\varphi(\ell) \cong \begin{cases} \mu\, \xi_{min}\, L\left[\frac{A^2}{2} \cos \omega_0 \ell + \nu^2 \delta(\ell)\right] + \frac{\mu A^4 L^2}{8\left[1 + \left(\frac{L}{2}\right) SNR\right]} \cos \omega_0 \ell \\ \quad + \frac{\mu\, \nu^2 A^2}{2\left[1 + \left(\frac{L}{2}\right) SNR\right]} (L - |\ell|) \cos \omega_0 \ell, \qquad |\ell| \leq L-1, & \text{(A.4)} \\ \left[\mu\, \xi_{min} \frac{LA^2}{2} + \frac{\mu A^4 L^2}{8\left[1 + \left(\frac{L}{2}\right) SNR\right]}\right] \cos \omega_0 \ell, \qquad |\ell| \geq L, & \text{(A.5)} \end{cases}$$

as the MF output autocorrelation. Note that the first term in (A.4) above is identical to (37). The second term represents an additional sinusoidal noise component in the MF output, while the last term represents a narrowband noise component having the same autocorrelation function form as the narrowband WF output noise [compare with (34a)].

In order to illustrate the relative magnitudes of these additional terms in the ALE output noise, we define two ratios R_δ and R_{NB} as follows. R_δ is the ratio of the power of the first sinusoidal noise term in (A.4) to that of the additional sinusoidal noise term which results from using (A.1) in place of (9). R_{NB} is the ratio of the power of the WF narrowband noise output to that of the additional narrowband noise term in the MF output. Then from (A.4) and (38a), we have

$$R_\delta = \frac{1}{a^*} + (1 - a^*)\, SNR_{in}, \tag{A.5}$$

$$R_{NB} = \frac{a^*}{\mu\, \xi_{min}\, L} . \tag{A.6}$$

Since $\mu\, \xi_{min}\, L \ll 1$ in practical applications, the latter ratio will greatly exceed unity over all the useful range of SNR_{in}. Thus the additional narrowband term resulting from (A.1) is of virtually no consequence. The ratio R_δ is infinite when SNR_{in} equals zero (since a^* is then zero), and it decreases monotonically to a value of unity as $a^* \to 1$ ($SNR_{in} \to \infty$). Thus the power in the additional sinusoidal noise term asymptotically approaches that of the first sinusoidal noise term for large enough values of SNR_{in}. This will reduce the ordinates of figures 4 and 5 by the amount

$$-10 \log_{10} \left(1 + \frac{1}{R_\delta}\right),$$

with a maximum reduction of -3.01 dB, as shown by the dotted curves in these figures. Although nonnegligible, the influence of this additional noise term is primarily in the high-gain region of the ALE, where both output sinusoidal noise terms are dwarfed by the sinusoidal signal output.

In summary, the intent of this appendix has been to show that the more precise weight covariance expression in [4] for the sinusoid in white noise output effects only minor modifications to the results obtained by using (9). Since (9) is applicable to more general inputs (for which the derivation in [4] would be intractable), we consider its use in our analysis to be justified.

REFERENCES

1. B. Widrow, J. R. Glover, Jr., J. M. McCool, J. Kaunitz, C. S. Williams, R. H. Hearn, J. R. Zeidler, E. Dong, Jr., and R. C. Goodlin, "Adaptive noise cancelling: Principles and Applications," Proc. IEEE, Vol. 63, pp. 1692–1716, Dec. 1975.

2. J. R. Zeidler and D. M. Chabries, "Analysis of the LMS adaptive filter used as a spectral line enhancer," Tech. Pub. TP 556, Naval Undersea Center, Feb. 1975.

3. J. R. Zeidler, E. H. Satorius, D. M. Chabries, and H. T. Wexler, "Adaptive enhancement of multiple sinusoids in uncorrelated noise," to appear in IEEE Trans. Acoust., Speech, Signal Processing.

4. J. R. Treichler, "The spectral line enhancer – The concept, an implementation, and an application," Ph.D. Disser., Dept. of Elec. Eng., Stanford Univ., June 1977.

5. B. Widrow, "Adaptive filters," in Aspects of Network and System Theory, R. Kalman and N. DeClaris, Eds., New York: Holt, Rinehart, & Winston, Inc., 1971.

6. A. V. Oppenheim and R. W. Schafer, Digital Signal Processing, Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1975.

7. L. J. Griffiths, J. Keeler, and R. Medaugh, "Detection and convergence results relating to the performance of an adaptive line enhancer," Tech. Note TN 1831, Naval Undersea Center, Dec. 1976.

8. S. T. Alexander, "The ALE output covariance function for a sinusoid in uncorrelated noise," Tech. Rpt. TR 162, Naval Ocean Systems Center, Aug. 1977.

9. S. T. Alexander, E. Satorius, and J. R. Zeidler, "Linear prediction and maximum entropy spectral analysis of finite-bandwidth signals in noise," to be presented at IEEE Conf. on Acoust., Speech, Signal Processing, Tulsa, OK, Apr. 10–12, 1978.